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Some Comments on the RHS Formulation of Resonance Scattering

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We discuss the validity of a relation between functionals in quantum resonance scattering which is often used in the current literature.

1. INTRODUCTION

This paper is a contribution to the theory of resonance scattering in which we discuss the validity of some formulas and concepts that appear in the current literature. These kinds of formulas are usually derived formally and used directly. Thus, an interpretation of them from the point of view of mathematical rigor is usually necessary to know their conditions of validity.

By resonance scattering we mean a scattering process that produces resonances [1–3]. We assume that this scattering process satisfies properties of regularity such as the existence of the Møller operators, asymptotic completeness, the absence of singular continuous spectrum, etc. Resonances are characterized by a pair of poles on the analytic continuation of the *S*-matrix beyond the cut on the positive semiaxis in the energy representation [1, 3, 4]. These pairs of poles have a real part which coincides with the resonance energy and an imaginary part which is the inverse of the mean lifetime. They are complex conjugates of each other. Some additional conditions on the behavior of the *S*-matrix on the second sheet are also imposed [5]. Under these conditions, we can separate the exponentially decaying part of a resonance from the background [2, 5]. This exponentially decaying part, also called a Gamov vector, is not a regular state in a Hilbert space and can be

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defined as a functional belonging to an extension of Hilbert space. This extension is given by a rigged Hilbert space (RHS).

Gamov vectors can be defined even if regularity conditions for the S-matrix are not present, but then the difficulty arises of the separation of the exponentially decaying part from the background. If the S-matrix poles are not simple, in addition to the exponentially decaying Gamov vectors, there must exist other functionals, also called Gamov vectors, for which the decay is a product of an exponential contribution times a polynomial in the time. These objects are so-called multiple-pole Gamov vectors, introduced elsewhere [6, 7]. Their energy representation has a Breit–Wigner part (as the exponentially decaying state vectors) plus other contributions [1].

The RHS formalism for resonances and Gamov vectors has been presented many times [1, 2, 5–9] and we refer the interested reader to the mentioned literature and the references quoted therein for this formalism. Here we use the notation of ref. 7 (see also [10]). Some of this material is repeated here in order to make this paper self-contained.

We start with the space of Hardy functions on the upper, $\mathcal{H}^2_+$, an lower, $\mathcal{H}^2_-$, half-planes [11] and we intersect them with the Schwartz space S. The restrictions of the functions on these spaces to the positive semiaxis give two locally convex nuclear spaces [5], which are denoted as $S \cap \mathcal{H}^2_\pm|_{\mathbb{R}^+}$ and their respective duals by $^\times(S \cap \mathcal{H}^2_\pm|_{\mathbb{R}^+})$. The triplets given by

$$S \cap \mathcal{H}^2_\pm|_{\mathbb{R}^+} \subset L^2(\mathbb{R}^+) \subset {}^\times(S \cap \mathcal{H}^2_\pm|_{\mathbb{R}^\pm}) \tag{1}$$

are RHS.

Next, we assume that the free Hamiltonian K and the total Hamiltonian $H = K + V$ have an absolutely continuous spectrum which is not degenerate and coincides with the positive semiaxis. The asumption of the nondegeneracy of K and H is not strictly necassary and we insert it in order to simplify the model and hence the notation. Under this circumstance, there is a unitary operator between the Hilbert space of the free states of our system and $L^2(\mathbb{R}^+)$, the space of the energy representation, that *diagonalizes* K, i.e., UKU^{-1} is the multiplication operator on $L^2(\mathbb{R}^+)$. If $\mathbf{\Phi}_\mp = U^{-1}(S \cap \mathcal{H}^2_\pm|_{\mathbb{R}^+})$, we have two new RHS:

$$\mathbf{\Phi}_\pm \subset \mathcal{H} \subset {}^\times\mathbf{\Phi}_\pm \tag{2}$$

where $\mathcal{H}$ represents the total Hilbert space if K has no eigenvectors. If K has eigenvectors, as in the Friedrichs model, we have to consider the absolutely continuous part of $\mathcal{H}$ with respect to K, which is defined as $\mathcal{H}_{ac}(K) := \mathcal{H} \ominus \mathcal{G}$. Here $\mathcal{G}$ is the space of bound states for K.

Since the Møller operators $\Omega_\pm$ exist and have the right properties, we can define $\Phi^\pm := \Omega_\pm \Phi_\pm$. We have new RHS:

$$\Phi^\pm \subset \mathcal{H}_{ac}(H) \subset {}^\times\Phi^\pm \tag{3}$$

It is precisely in the space ${}^\times\Phi^\pm$ where the Gamov vectors live.

The purpose of this paper is to discuss some formulas and constructions that sometimes appear in the literature [12–15] and that find their own motivation precisely in the RHS formulation of resonance scattering.

2. PRESENTATION

We start with a formula which is often shown and discuss its validity. The point of departure is the RHS given in (3) with a plus sign. Let $\varphi^+ \in \Phi^+$. The theorem of Gelfand and Maurin establishes the existence of a complete set of generalized eigenvectors of the total Hamiltonian H such that, $\forall\varphi^+ \in \Phi^+$,

$$\varphi^+ = \int_0^\infty dE\, |E^+\rangle\langle^+E|\varphi^+\rangle \tag{4}$$

and $H|E^+\rangle = E|E^+\rangle$. On the other hand, take $\psi^- \in \Phi^-$ and $\varphi^+ \in \Phi^+$. We have the following formula [1, 2, 5]:

$$(\psi^-, \varphi^+) = \int_0^\infty dE\, \langle\psi^-|E^-\rangle S(E + i0)\langle E^+|\varphi^+\rangle \tag{5}$$

where the functionals $|E^-\rangle \in {}^\times\Phi^-$ are also generalized eigenvectors of H. From (4) and (5), one gets

$$\varphi^+ = \int_0^\infty dE\, |E^-\rangle S(E + i0)\langle E^+|\varphi^+\rangle \tag{6}$$

Comparing (4) with (6), one is tempted to conclude that

$$|E^-\rangle S(E + i0) = |E^+\rangle \tag{7}$$

This is the first point on which we want to comment. First, we have to realize that formulas (4) and (6) do not refer to the same kind of object. In (4), φ^+ represents the unique element of ${}^\times\Phi^+$ which is the image of $\varphi^+ \in \Phi^+$ into ${}^\times\Phi^+$ by the natural imbedding. In (6), it is an element of ${}^\times\Phi^-$ which does not come from Φ^-, but from the Hilbert space (which contains Φ^+). Therefore, they are functionals on two different spaces. In addition, the functionals $|E^-\rangle$ and $|E^+\rangle$ act on two different spaces. Therefore, they cannot be proportional to each other.

Nevertheless, one may think that the functionals $|E^-\rangle$ and $|E^+\rangle$ could be extended into the space $\Phi = \Phi^+ + \Phi^-$ and then find a possible relation between them. The space Φ can be endowed with a topology using the topologies on Φ^+ and Φ^-.

In general, topologies on a locally convex space are given by a family of seminorms. Seminorms have the same property as norms with the exception that the seminorm of a nonzero vector may be zero. In particular, norms are seminorms. If a vector space Φ is the *direct* sum of two locally convex spaces, $\Phi = \Phi_1 \oplus \Phi_2$, we can construct a locally convex topology on it defining seminorms. In fact any $\varphi \in \Phi$ can be uniquely written as $\varphi = \varphi_1 + \varphi_2$, where $\varphi_1 \in \Phi_1$ and $\varphi_2 \in \Phi_2$. Then, if p_1 is a seminorm on Φ_1 and p_2 is a seminorm on Φ_2, $p(\varphi) = p_1(\varphi_1) + p_2(\varphi_2)$ defines a seminorm on Φ and all seminorms on Φ are defined in this manner.

However, if $\Phi_1 \cap \Phi_2$ is nontrivial, a vector of Φ cannot be written in a *unique* manner as the sum of a vector of Φ_1 plus a vector of Φ_2, but there are infinitely many choices in general. For this reason, the seminorms in the sum space Φ cannot be written as above. It is nevertheless true that for a seminorm p_1 on Φ_1 and a seminorm p_2 on Φ_2 we have a unique seminorm p on Φ, which is defined as

$$p_{12}(\varphi) = \inf\{p_1(\varphi_1) + p_2(\varphi_2)\} \tag{8}$$

where the infimum is taken over all possible forms of decomposing $\varphi \in \Phi$ as a sum of a vector φ_1 in Φ_1 and a vector φ_2 in Φ_2.

In our case, this nonuniqueness is a bad property, and contributes to make useless the space $\Phi := \Phi^+ + \Phi^-$. Let us show why. One may think that the vectors $|E^+\rangle$ and $|E^-\rangle$ could be extended to this sum and then they could be compared. This is false for simple algebraic reasons. The problem arises because the operations that carry Φ^+ and Φ^- to spaces of Hardy functions are different. This operation is a product of a Møller operator times a unitary operator and we use a different Møller operator in each case. Thus, if $\varphi = \varphi^+ + \varphi^-$ and $\varphi_-(E)$ is the Hardy function corresponding to φ^+ and $\varphi_+(E)$ the Hardy function corresponding to φ^-, the function $\varphi(E) = \varphi_-(E) + \varphi_+(E)$ is well defined although it depends on the way that we decompose φ as a sum of an element of Φ^+ and an element of Φ^-. To show this, let us take $\varphi \in \Phi^+ \cap \Phi^-$. As a vector in Φ^+, φ is represented by the function $\varphi_-(E) = U\Omega_+^{-1}\varphi$; as a vector in Φ^-, φ is represented by the function $\varphi_+(E) = U\Omega_-^{-1}\varphi$; these are different in general. Thus, the mapping given by

$$\varphi \mapsto \varphi(E) \tag{9}$$

is not well defined. Unfortunately, this is the only serious candidate to extend simultaneously $|E^+\rangle$ and $|E^-\rangle$.

However, we can give meaning to formula (7) on a smaller space. Let us take the intersection $\Phi^+ \cap \Phi^-$ and assume that it is nontrivial. Take $\varphi \in \Phi^+ \cap \Phi^-$. This φ can be viewed either as an element of Φ^+, and then we call it φ^+, or as an element of Φ^-, and then we call it φ^-. Let us consider the following pair of functions:

$$\langle E^+|\varphi^+\rangle = U\Omega_+^{-1}\varphi; \qquad \langle E^-|\varphi^-\rangle = U\Omega_-^{-1}\varphi \tag{10}$$

From (10), we can see that

$$\begin{aligned}\langle E^+|\varphi^+\rangle &= U\Omega_+^{-1}\Omega_- U^{-1}\langle E^-|\varphi^-\rangle \\ &= US^{-1}U^{-1}\langle E^-|\varphi^-\rangle \\ &= S^*(E+i0)\langle E^-|\varphi^-\rangle \end{aligned}\tag{11}$$

This implies that

$$\langle E^+| = S^*(E+i0)\langle E^-| \Leftrightarrow |E^+\rangle = S(E+i0)|E^-\rangle \tag{12}$$

but this formula is valid only when both terms of the identity act on vectors of $\Phi^+ \cap \Phi^-$.

A similar situation happens with the Gamov vectors. One candidate for the extension of $|f_0\rangle$ to Φ is the following:

$$\varphi = \varphi^+ + \varphi^-; \qquad \text{then} \quad \langle\varphi|f_0\rangle := \langle\varphi^-|f_0\rangle \tag{13}$$

Since the decomposition is not unique, $|f_0\rangle$ is not well defined on φ.

Once we have raised the question whether there is a relation between $|E^+\rangle$ and $|E^-\rangle$, it becomes of interest to answer it. Take $E_0 > 0$. Consider the Schwartz space S and the distribution $\delta^*(E - E_0)$ defined as

$$\int_{-\infty}^{\infty} f(E)\delta^*(E-E_0)\, dE = f^*(E_0) \tag{14}$$

where $f^*(E_0)$ is the complex conjugate of the value of the Schwartz function $f(E)$ at the point E_0. This distribution is an antilinear continuous functional on S and can be viewed as the complex conjugate of the Dirac delta. Now, take the subspace given by the direct sum

$$(\mathcal{H}_+^2 \cap S) \oplus (\mathcal{H}_-^2 \cap S) \tag{15}$$

This is a proper subspace of S and therefore the functional $\delta^*(E - E_0)$ can be restricted to it, as it can be restricted to both $\mathcal{H}_+^2 \cap S$ and $\mathcal{H}_-^2 \cap S$. Following the definitions of $\Phi_\pm$ and $\Phi^\pm$, there exist two unitary mappings $V_\pm$

$$V_\pm = U\Omega_\pm^{-1} \tag{16}$$

such that

$$V_\pm\Phi^\pm = \mathcal{H}_\mp^2 \cap S|_{\mathbb{R}^+} = \theta_\mp(\mathcal{H}_\mp^2 \cap S) \tag{17}$$

where $\theta_\mp$ gives the relation between a function on $\mathcal{H}_\mp^2$ and its restriction to $\mathbb{R}^+$. This relation is one to one, due to the van Winter theorem [17, 5].

By duality, we can extend all these operators to relations between the dual spaces, and the relations (16) hold for the extended operators. In particular,

$$V_{\pm}\ {}^{\times}\Phi^{\pm} = U\Omega_{\pm}^{-1}\ {}^{\times}\Phi^{\pm} = {}^{\times}(\mathcal{H}^2_{\mp} \cap S|_{\mathbb{R}^+}) = \theta_{\mp}^{\times}[{}^{\times}(\mathcal{H}^2_{\mp} \cap S)] \quad (18)$$

After these comments, we can see [5] that for any fixed value $E_0 \in \mathbb{R}$ and the definition of $|E_0^{\pm}\rangle$ ($\langle\varphi^{\pm}|E_0^{\pm}\rangle = [\varphi^{\mp}(E_0)]^*$; $\varphi^{\mp}(E) = \theta_{\mp}^{-1}V_{\pm}\varphi^{\pm}$; $\forall\varphi^{\pm} \in \Phi^{\pm}$), that:

$$|E_0^+\rangle = \Omega_+ U^{-1}\theta_-^{\times}[\delta^*(E - E_0)] \quad (19)$$

$$|E_0^-\rangle = \Omega_- U^{-1}\theta_+^{\times}[\delta^*(E - E_0)] \quad (20)$$

This equations give

$$|E_0^+\rangle = \Omega_+ U^{-1}\theta_-^{\times}(\theta_+^{\times})^{-1}U\Omega_-^{-1}|E_0^-\rangle \quad (21)$$

We observe that the operator that relates these two kets depends (i) on the *dynamics* through the Møller wave operators, and (ii) on *causality* via the properties of Hardy functions through the $\theta_{\pm}$ operators. Nevertheless, its interesting to point out that the relation between them is not of the type

$$|E^+\rangle = (\Omega_-)^{-1}O\Omega_+|E^-\rangle$$

where O is an operator, as suggested by (7) [recall that $S = (\Omega_-)^{-1}\Omega_+$].

Summarizing, formula (7) *does not make sense* on the space $\Phi = \Phi^+ + \Phi^-$.

In any case, to make sense out of formula (7), we need to find a space in which both $|E^-\rangle$ and $|E^+\rangle$ act. This will necessarily imply a redefinition of either Φ^+ or Φ^- or both. On the other hand, it seems natural that the space of outgoing states be the result of the action of the S-operator on the space of incoming states, so that

$$S\Phi_+ = \Phi^{\text{out}} \Leftrightarrow \Omega_+\Phi_+ = \Omega_-\Phi^{\text{out}} \quad (22)$$

where Φ_+ is defined as in (2). Here, $|E^-\rangle$ is a functional on $\Phi^- = \Omega_-\Phi^{\text{out}}$ that should be defined as in the standard case [5]. Take $\varphi^- \in \Phi^-$ and the function given by $\varphi_+(E) = U\Omega_-^{-1}\varphi^-$. This function is defined on $\mathbb{R}^+$. Then, if $E_0 < 0$, $\langle\varphi^-|E_0^-\rangle$ is given by the value of the function $[\varphi_+(E)]^*$ at the point E_0. The meaning of $|E^+\rangle$ does not change as a functional on $\Omega_+\Phi_+$. Since these spaces are equal, $|E^-\rangle$ and $|E^+\rangle$ may be compared. To do this, let us choose $\varphi^{\text{in}} \in \Phi_+$ and $\psi^{\text{out}} \in \Phi^{\text{out}}$. Then, $\psi^- = \Omega_-\psi^{\text{out}}$, $\varphi^+ = \Omega_+\varphi^{\text{in}}$; we have

$$\langle\varphi^+|E^+\rangle = \langle\Omega_+\varphi^{\text{in}}|\Omega_+|E\rangle = \langle\varphi^{\text{in}}|E\rangle = [\langle E|\varphi^{\text{in}}\rangle]^*$$
$$= [U\varphi^{\text{in}}]^*(E) = [\varphi^-(E)]^* \in \mathcal{H}^2_+ \cap S|_{\mathbb{R}^+} \quad (23)$$

where $K|E\rangle = |E\rangle$ for $E > 0$, (see discussion in the last part of the paper on the Dirac kets $|E\rangle$ for the free Hamiltonian K). The action of $|E^-\rangle$ on an arbitrary $\varphi^+ \in \Phi^+$ is defined as

$$\langle\varphi^+|E^-\rangle = \langle\Omega_+\varphi^{\text{in}}|\Omega_-|E\rangle = \langle\Omega_-^\dagger\Omega_+\varphi^{\text{in}}|E\rangle$$
$$= \langle S\varphi^{\text{in}}|E\rangle = [S(E+i0)\varphi^-(E)]^* = S^*(E+i0)\langle\varphi^+|E^+\rangle \quad (24)$$

Thus,

$$|E^-\rangle = S^*(E+i0)|E^+\rangle \Leftrightarrow |E^+\rangle = S(E+i0)|E^-\rangle \quad (25)$$

since $S(E+i0)$ for $E > 0$ is a complex number with modulus equal to one. This procedure has some inconveniences:

1. The function

$$\langle\psi^-|E^-\rangle S(E+i0)\langle E^+|\varphi^+\rangle \quad (26)$$

is the function that appears under the integral sign in the explicit expression for $(\psi^{\text{out}}, S\varphi^{\text{in}})$. This scalar product is the point of departure of the construction of Gamov vectors à la Bohm and the separation between the contribution of the Gamov vectors and the background integral to the decaying process [1, 5]. The function (26) is not meromorphic on the lower half-plane, since $\langle\psi^-|E^-\rangle \in \mathcal{H}_+^2 \cap S|_{\mathbb{R}^*}$. To save this situation, we may choose $\psi^{\text{out}} \in \Phi_+ \cap \Phi_-$, so that $\langle\psi^-|E^-\rangle \in \mathcal{H}_+^2 \cap S|_{\mathbb{R}^+} \cap \mathcal{H}_-^2 \cap S|_{\mathbb{R}^+}$. This intersection is not trivial [9], but we do not know whether it is dense or not. Even if this intersection were dense, this consideration will create problems in defining the time evolution for the Gamov vectors, as we shall see.

2. Worst of all, the expression $(\psi^{\text{out}}, S\varphi^{\text{in}})$ becomes totally useless for the theory of resonances. In fact, since $\psi^{\text{out}} \in S\Phi_+$, there exists $\psi^{\text{in}} \in \Phi_+$ such that $\psi^{\text{out}} = S\psi^{\text{in}}$. Thus, $(\psi^{\text{out}}, S\varphi^{\text{in}}) = (S\psi^{\text{in}}, S\varphi^{\text{in}}) = (\psi^{\text{in}}, \varphi^{\text{in}})$, which is independent of the scattering process and therefore does not show resonances.

We could pose the problem in other terms: Is it possible to write a formula like

$$S|E^-\rangle = |E^+\rangle? \quad (27)$$

In order to do it, it seems clear that we should extend S to ${}^\times\Phi^-$ first. Let us analyze all possibilities. First let us take the vector space given by

$$S\Phi^- = \{\eta \in \mathcal{H}_{ac}(H)/\exists\varphi^- \in \Phi^-\colon \eta = S\varphi^-\} \quad (28)$$

where S is the S-operator on $\mathcal{H}$. For simplicity, we shall assume that $S\mathcal{H}_{ac}(H) = \mathcal{H}_{ac}(H)$, which, under the posed conditions, happens, for instance, if H has no Hilbert space eigenvalues.

Then, S can be extended to the dual ${}^{\times}\Phi^{-}$ by duality, so that we have a new RHS:

$$S\Phi^{-} \subset \mathcal{H}_{ac}(H) \subset {}^{\times}[S\Phi^{-}] = S({}^{\times}\Phi^{-}) \tag{29}$$

In order to compare the functionals $S|E^{-}\rangle$ and $|E^{+}\rangle$, they should act on the same space. If $S|E^{-}\rangle$ acts on Φ^{+}, the bracket $\langle\varphi^{+}|S|E^{-}\rangle$ must be well defined for any $\varphi^{+} \in \Phi^{+}$. This means that for all $\varphi^{+} \in \Phi^{+}$, $\exists\psi^{-} \in \Phi^{-}$ such that $S\psi^{-} = \varphi^{+}$, i.e., $\Phi^{+} \subset S\Phi^{-}$. Also, if $|E^{+}\rangle$ acts on $S\Phi^{-}$, for any $\psi^{-} \in \Phi^{-}$, then the bracket $\langle S\psi^{-}|E^{+}\rangle$ makes sense. Thus, $S\psi^{-} \in \Phi^{+}$, $\forall\psi^{-} \in \Phi^{-}$, which means that $S\Phi^{-} \subset \Phi^{+}$. Consequently,

$$S\Phi^{-} = \Phi^{+} \tag{30}$$

This implies that for any $\varphi^{+} \in \Phi^{+}$ there exists $\psi^{-} \in \Phi^{-}$ such that $S\psi^{-} = \varphi^{+}$ and vice versa. This identity has the following equivalent forms:

$$\begin{aligned}\varphi^{+} = S\psi^{-} &\Leftrightarrow \Omega_{+}\varphi_{+} = S\Omega_{-}\psi_{-} \Leftrightarrow \Omega_{+}U^{-1}\varphi_{-}(E)\\ &= S\Omega_{-}U^{-1}\psi_{+}(E) \Leftrightarrow \varphi_{-}(E) = U\Omega_{+}^{\dagger}S\Omega_{-}U^{-1}\psi_{+}(E)\\ &= U\Omega_{+}^{\dagger}U^{-1}USU^{-1}U\Omega_{-}U^{-1}\psi_{+}(E)\end{aligned} \tag{31}$$

Since $USU^{-1} = S(E + i0)$, (31) is equal to

$$\varphi_{-}(E) = S(E + i0)U\Omega_{+}^{\dagger}\Omega_{-}U^{-1}\psi_{+}(E) = S(E + i0)US^{-1}U^{-1}\psi_{+}(E) \tag{32}$$

Analogously,

$$US^{-1}U^{-1}\psi_{+}(E) = S^{*}(E + i0)\psi_{+}(E)$$

With the property that $S(E + i0)$, with $E > 0$, is a complex number with modulus one, we finally have

$$\langle E^{+}|\varphi^{+}\rangle = \varphi_{-}(E) = \psi_{+}(E) = \langle E^{-}|\psi^{-}\rangle \tag{33}$$

Formula (33) does not make sense if we keep for $\Phi^{\pm}$ the definitions given in (16). In fact, (33) means that $V_{+}\Phi^{+} = V_{-}\Phi^{-}$, which is not true following (16). Therefore, in order to make formula (27) mathematically meaningful, we need to reconstruct $\Phi^{\pm}$ with the mentioned property that $V_{+}\Phi^{+} = V_{-}\Phi^{-}$. This is what we are going to do in the next few paragraphs.

To begin with, let us consider now the space $S(\mathbb{R}^{-})$ of all Schwartz functions supported in the negative semiaxis $\mathbb{R}^{-} = (-\infty, 0]$. For any $f \in S(\mathbb{R}^{-})$, the function $\mathfrak{H}f \pm if \in \mathcal{H}^{2}_{\pm}$, where $\mathfrak{H}$ is the Hilbert transform [16].

Proposition. The Fourier transform $\mathcal{F}(\mathfrak{H}f \pm if)$ is the restriction to $\mathbb{R}^{\mp}$ of a Schwartz function.

Proof. Being given a function f, its Hilbert transform $\mathfrak{H}f$ represents the convolution of f with the distribution $PV(1/x)$ (PV denotes Cauchy principal value). We know that

$$2PV\left(\frac{1}{x}\right) = \frac{1}{x+i0} + \frac{1}{x-i0} \tag{34}$$

The Fourier transform of this distribution is given by

$$\mathcal{F}\left(PV\left(\frac{1}{x}\right)\right) = i(H(-x) - H(x)) = iG(x) \tag{35}$$

where $H(x)$ is the Heaviside or step function. It is zero on the negative semiaxis and one on the positive semiaxis. Since the Fourier transform of the convolution of two functions is the product of the Fourier transforms of both functions, we conclude that

$$\mathcal{F}(\mathfrak{H}f \pm if) = iG(x)\hat{f}(x) \pm i\hat{f}(x)\,, \qquad \text{where} \quad \hat{f} = \mathcal{F}(f) \tag{36}$$

This Fourier transform is explicitly given as follows

If $x > 0$,

$$\mathcal{F}(\mathfrak{H}f + if) = 0; \qquad \mathcal{F}(\mathfrak{H}f - if) = -2i\hat{f}(x)|_{\mathbb{R}^+} \tag{37}$$

If $x < 0$,

$$\mathcal{F}(\mathfrak{H}f + if) = 2i\hat{f}(x)|_{\mathbb{R}^-}; \qquad \mathcal{F}(\mathfrak{H}f - if) = 0 \tag{38}$$

Equations (37) and (38) give us the desired result. ■

The functions which are restrictions of Schwartz functions to $\mathbb{R}^\pm$ are dense in $L^2(\mathbb{R}^\pm)$. Thus we conclude by the Paley–Wienner theorem [11] that

$$(\mathfrak{H}f \pm if)S(\mathbb{R}^-)$$

is dense in $\mathcal{H}^2_\pm$.

As a consequence of the van Winter theorem [17, 5], the restrictions of a function in $\mathcal{H}^2_\pm$ to either $\mathbb{R}^+$ or $\mathbb{R}^-$ determine all the values of the function. Furthermore, these restrictions are dense in $L^2(\mathbb{R}^+)$ and $L^2(\mathbb{R}^-)$. Since the functions in

$$(\mathfrak{H}f \pm if)S(\mathbb{R}^-)$$

are dense in $\mathcal{H}^2_\pm$, it immediately follows that their restrictions to $L^2(\mathbb{R}^+)$ are dense in this space.

Furthermore, the restrictions of the functions in $(\mathfrak{H}f \pm if)S(\mathbb{R}^-)$ to $\mathbb{R}^+$ satisfy an important property: they admit a unique extension to $\mathbb{R}^-$ to be in $\mathcal{H}^2_+$ and another unique extension to $\mathbb{R}^-$ to be in $\mathcal{H}^2_-$ and these extensions are different, since the intersection $\mathcal{H}^2_+ \cap \mathcal{H}^2_- = \{0\}$. To see this, recall that any $f \in S(\mathbb{R}^-)$ is zero on the positive semiaxis, so that $\mathfrak{H}f \pm if|_{\mathbb{R}^+} = \mathfrak{H}f|_{\mathbb{R}^+}$. Thus, $\mathfrak{H}f \pm if \in \mathcal{H}^2_\pm$, but are indistinguisable on the positive semiaxis.

Now, let us define the following space:

$$\Delta = \{g \in L^2(\mathbb{R}^+), \text{ such that } \exists f \in S(\mathbb{R}^-) \text{ with } g = \mathfrak{H}f \pm if|_{\mathbb{R}^+}\} \quad (39)$$

As we have seen, Δ is dense in $L^2(\mathbb{R}^+)$. The space of the restrictions to $\mathbb{R}^-$ of Schwartz functions can be endowed with a nuclear locally convex topology exactly as we do on the Schwartz space. Then, it is immediate to show that

$$\Delta \subset L^2(\mathbb{R}^+) \subset {}^\times\Delta \quad (40)$$

is a RHS. If we construct

$$\Phi^\pm = V_\pm^{-1}\Delta \quad (41)$$

we have triplets with the required conditions so that the formula $S|E^-\rangle = |E^+\rangle$ is correctly defined. Note that in this case we have for the spaces defined right before (2) that $\Phi_+ = \Phi_- = U^{-1}\Delta$.

It seems that we have found triplets that satisfy all possible good properties. But this is false! Certainly, we can define the Gamow vectors using the triplet (3) with $\Phi^\pm$ defined as in (41), but this causes a severe difficulty: if $\varphi(E) \in \Delta$, then $e^{itE}\varphi(E) \notin \Delta$ for any $t \neq 0$. The reason is very simple: assume, for instance, that $t > 0$. For $f(E) \in \Delta$, the integrals

$$\int_{-\infty}^{\infty} |e^{it(E+i\alpha)}f(E + i\alpha)|^2 \, dE = e^{-t\alpha} \int_{-\infty}^{\infty} |f(E + i\alpha)|^2 \, dE \quad (42)$$

are uniformly bounded for $\alpha > 0$, but they *are not* for $\alpha < 0$. This means that there is *no time evolution* for $\Phi^\pm$ and its duals, when they are defined as in (41), because then, for any $t \neq 0$ and any $\varphi^\pm \in \Phi^\pm$, $e^{itH}\varphi^\pm \notin \Phi^\pm$.

Summarizing. If we want to give meaning to formula (7), we lose information about S-matrix poles. If we want to give meaning to formula (27), we lose the time behavior for Gamov vectors. ■

We want to end this paper with another comment. Let us consider again the spaces $\Phi_\pm$ and the RHS given in (2). We know that the free Hamiltonian K reduces these spaces ($K\Phi_\pm \subset \Phi_\pm$) and is continuous on them. Therefore, it can be continuously extended to $\Phi_\pm^\times$ by duality. Since the spectrum of K coincides with the positive semiaxis $\mathbb{R}^+$, for any $E \in \mathbb{R}^+$ there is a generalized eigenvector $|E_\pm\rangle$ of K in $\Phi_\pm^\times$: $K|E_\pm\rangle = E|E_\pm\rangle$. These functionals are defined as follows: Let $\varphi_\pm \in \Phi_\pm$ and $U\varphi_\pm = \varphi_\mp(E)$. Take $E' \in \mathbb{R}^+$; the functional $|E'_\pm\rangle$ is defined by the mapping

$$\varphi_\pm \mapsto [\varphi_\mp(E')]^* = \langle\varphi_\pm|E'_\pm\rangle \tag{43}$$

We want to show that $|E_\pm\rangle$ are indeed the restrictions to $\Phi_\pm$ of a functional on a bigger space. This bigger space is

$$\Psi = \Phi_+ + \Phi_- \tag{44}$$

Let us take $\varphi \in \Psi$. Then, $\varphi = \varphi_+ + \varphi_-$ and this decomposition is not unique. Let us define the action of U on $\varphi \in \Psi$ as $\varphi(E) := U\varphi = U\varphi_+ + U\varphi_- = \varphi_-(E) + \varphi_+(E)$. Obviously, this mapping is well defined and does not depend on the decomposition of φ. For any $E \in \mathbb{R}^+$, the mapping on Ψ given by $\varphi \mapsto [\varphi(E)]^* = \langle\varphi|E\rangle$ is well defined. To prove its continuity, we write

$$\begin{aligned}|\varphi(E)| \leq |\varphi_-(E)| + |\varphi_+(E)| &\leq \sup\{|\varphi_-(E)|\} + \sup\{|\varphi_+(E)|\} \\ &= p_0(\varphi_+) + p_0(\varphi_-)\end{aligned} \tag{45}$$

Here, p_0 represents one of the seminorms (here are indeed norms) that provide the topology on $\Phi_\pm$ [18]. Since the above inequality holds for any decomposition of φ, we finally have

$$|\varphi(E)| \leq \inf\{p_0(\varphi_+) + p_0(\varphi_-)\} \tag{46}$$

proving the desired continuity. The restriction of the functional $|E\rangle$ to $\Phi_\pm$ is $|E_\pm\rangle$. This justifies the unique notation as $|E\rangle$ for these two functionals that appear in some publications [1].

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